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ACTIONS OF HYPERBOLIC THREE-MANIFOLD GROUPS ON COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper we describe a discontinuity domain for the natural action of hyperbolic three-manifold groups on complex projective spaces of arbitrary dimension.

1. INTRODUCTION

In recent years the study of representations of hyperbolic three-manifold groups into $SL_n(\mathbf{C})$ is playing an important rôle. Among others, we mention the work of Werner Müller [17], Jonathan Pfaff [25, 24], W. Müller and J. Pfaff, [19, 18, 20], Stavros Garoufalidis, Dylan Thurston, and Christian Zickert [9], S. Garoufalidis, Matthias Görner, and C. Zickert [8], Takashi Hara and Takahiro Kitayama, and Pere Menal-Ferrer and myself [16].

There is a distinguished representation in $SL_{n+1}(\mathbf{C})$ constructed as follows. We start with the definition of symmetric power. Consider $\mathbf{C}[X, Y]$ the algebra of polynomials on two variables. We have a natural action of $SL_2(\mathbf{C})$ on $\mathbf{C}[X, Y]$ by precomposition

$$\begin{array}{ccc} SL_2(\mathbf{C}) \times \mathbf{C}[X,Y] & \to & \mathbf{C}[X,Y] \\ A, P & \mapsto & P \circ A^t \end{array}$$

where A^t denotes the transpose of A. Notice that transposing or taking the inverse in $PSL_2(\mathbf{C})$ differ by conjugation by a matrix, thus the action $P \mapsto P \circ A^{-1}$ is equivalent. This action restricts to the homogeneous polynomials of degree n, which define a $n+1$ dimensional subspace of $\mathbf{C}[X, Y]$:

$$\mathbf{C}_n[X,Y] = \{p(X,Y) \in \mathbf{C}[X,Y] \mid p \text{ is homogeneous and } \deg(p) = n\}.$$

Definition 1.1. The *n-symmetric representation*

$$\mathrm{Sym}_n : SL_2(\mathbf{C}) \to SL_{n+1}(\mathbf{C})$$

is defined by the action on homogeneous polynomials on two variables of degree n.

Let M^3 be a closed, compact, hyperbolic and orientable three-manifold. Fix a lift of its holonomy representation

$$\widetilde{\mathrm{hol}} : \pi_1(M^3) \to SL_2(\mathbf{C}).$$

We consider then the representation

$$\rho_n = \pi \circ \mathrm{Sym}_n \circ \widetilde{hol} : \pi_1(M^3) \to SL_{n+1}(\mathbf{C}) \to PSL_{n+1}(\mathbf{C}), \tag{1}$$

where $\pi : SL_{n+1}(\mathbf{C}) \to PSL_{n+1}(\mathbf{C})$ is the natural projection. Notice that ρ_n does not depend on the lift. This induces a natural action of $\pi_1(M^3)$ on complex projective space $\mathbf{P}^n$ but also on the flag manifolds of $\mathbf{P}^n$.

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Question 1.2. *Find a domain $X_n \subset \mathbf{P}^n$ (or in a flag manifold of $\mathbf{P}^n$) such that the action of $PSL_2(\mathbf{C})$ induced by Sym_n is proper and, if possible, cocompact. Describe the quotients $PSL_2(\mathbf{C})\backslash X_n$ and $\rho_n(\pi_1(M^3))\backslash X_n$.*

The question for surfaces has been addressed by Guichard and Weinhart, with the so called Anosov representations [10]. In our case, when M is compact, ρ_n is also an Anosov representation.

Here we answer Question 1.2 by finding a domain in complex projective space. For the dynamics of discrete groups in complex projective space, see also the work of Cano, Navarrete and Seade in [3] and references therein. This is also addressed in a more general setting in a joint project with Misha Kapovich and Bernhard Leeb, as $\mathbf{P}^n$ and flag manifolds appear in the Tits boundary of symmetric spaces of nonpositive curvature.

We mention that Sym_1 is the identity, and that ρ_1 is just the lift of the holonomy representation. In this case there is no proper action on $\mathbf{P}^1$. The case $n = 2$ will be addressed in Section 2, by considering the flag manifold. When $n \geq 3$, we will find a domain in complex projective space $\mathbf{P}^n$.

For $n \geq 3$, we deal with an invariant curve and the osculating variety. We start with the *Veronese* embedding

$$\begin{array}{ccc} \mathbf{P}^1 & \to & \mathbf{P}^n \\ (a:b) & \mapsto & (aX+bY)^n \end{array} \tag{2}$$

Its image $Q_n \subset \mathbf{P}^n$ is an algebraic curve (isomorphic to $\mathbf{P}^1$) invariant under the action of $\mathrm{Sym}_n(PSL_2(\mathbf{C}))$, called the *rational normal curve* [7]. The action on $\mathbf{P}^n - Q_n$ is still not proper. For this we shall remove a larger subset of the osculating manifold. Recall that an affine k-plane is osculating to a curve if at one point it contains all derivatives of order $\leq k$. This is an affine notion that generalizes to the projective setting.

Definition 1.3. The *k-osculating variety* to Q_n is the set of projective k-planes that are k-osculating to Q_n and it is denoted by $Osc_k(Q_n)$.

For all k, $Osc_k(Q_n)$ is invariant by the action of $\mathrm{Sym}_n(PSL_2(\mathbf{C}))$. The good choice will be $k = [n/2]$, the integer part of $n/2$.

Theorem 1.4. *For $n > 2$, the action of $\mathrm{Sym}_n(PSL_2(\mathbf{C}))$ is proper on*

$$X_n = \mathbf{P}^n - Osc_{[n/2]}(Q_n).$$

For n odd, the quotient $PSL_2(\mathbf{C})\backslash X_n$ is a smooth complex projective variety. For n even, the quotient $PSL_2(\mathbf{C})\backslash X_n$ admits a natural one point compactification which is a complex projective variety, smooth for $n = 4$ and with precisely a singular point for $n > 4$.

Since $\pi_1(M^3)\backslash PSL_2(\mathbf{C})$ is the frame bundle of M^3, we have the following corollary.

Corollary 1.5. *Let M^3 be an orientable and hyperbolic three-manifold. Then the quotient $\rho_n(\pi_1(M^3))\backslash X_n$ is a smooth complex variety that fibres over M^3 and also over its frame bundle (except when $n = 3$). The fiber is compact for n odd, and for n even it admits a compactification that consists in adding a point for each fibre of the frame bundle.*

The exception when $n = 3$ is that it is the quotient of the frame bundle by the action of the permutation group on three elements (i.e. the bundle of *unordered* frames).

The paper is organized as follows. In Section 2 we discuss first the action of *Sym^2* on the flag manifold. Notice that the action on $\mathbf{P}^2$ cannot be proper because of dimensions.

Then in Section 3 we prove properness and cocompactness by using standard methods of hyperbolic geometry, namely the the barycenter for configurations of ideal points. To prove that the quotient (or its one point compactification) is a complex projective manifold, we use geometric invariant theory in Section 4, as this example was precisely computed in Mumford's book [21]. Then in Section 5 we establish smoothness of the quotient and nonsmoothness of its compactification, which is probably the only new result of the paper. Finally, Section 6 is devoted to compute explicitly some low dimensional examples.

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2. The action of Sym_2

Theorem 1.4 only applies for $n \geq 3$. We discuss first $n = 2$ as an exceptional low dimensional case. Notice that $PSL_3(\mathbf{C})$ acts naturally on the projective space $\mathbf{P}^2$, so the stabilizer of a point in $\mathbf{P}^2$ of the action of $\mathrm{Sym}_2(PSL_2(\mathbf{C}))$ is a complex manifold of dimension at least one, hence it cannot be proper. To find proper actions we shall work in the flag manifold.

Definition 2.1. The *flag manifold* of $\mathbf{P}^2$ is the set of pairs (p, L) where p is a line in $\mathbf{C}^3$ (a point in $\mathbf{P}^2$) and L a plane in $\mathbf{C}^3$ (a line in $\mathbf{P}^2$) containing p. It is denoted by $F(2)$.

If $(\mathbf{P}^2)^*$ denotes the dual to $\mathbf{P}^2$, then

$$F(2) = \{(p, L) \in \mathbf{P}^2 \times (\mathbf{P}^2)^* \mid p \in L\}.$$

Using homogeneous coordinates for the points $p = [x_1 : x_2 : x_3]$ and writing the elements of $(\mathbf{P}^2)^*$ also with homogeneous coordinates $L = [a_1 : a_2 : a_3]$ corresponding to the line defined by the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, we have the following remark.

Remark 2.2. The flag manifold $F(2)$ is isomorphic to the hypersurface

$$\{([x_1 : x_2 : x_3], [a_1 : a_2 : a_3]) \in \mathbf{P}^2 \times \mathbf{P}^2 \mid x_1a_1 + x_2a_2 + x_3a_3 = 0\}.$$

In particular it is three-dimensional

Thus $F(2)$ has already the right dimension to find a domain where the action is proper and cocompact. To find such a domain, we must consider and invariant subset. More precisely, $\mathbf{P}^2$ is the projective space on the vector space of homogeneous quadratic polynomials

$$p(X, Y) = aX^2 + bXY + cY^2$$

Consider the quadric Q_2 defined by the polynomials that have a double root; namely the polynomials with zero discriminant:

$$Q_2 = \{aX^2 + bXY + cY^2 \in \mathbf{C}_2[X, Y] \mid b^2 - 4ac = 0\}.$$

The quadric Q_2 is isomorphic to $\mathbf{P}^1$ and it is invariant by the action of $PSL_2(\mathbf{C})$. It is in fact the *rational normal curve* of the introduction, the image of the Veronese embedding (2). The main result for $n = 2$ is the following:

Theorem 2.3. *Viewing the flag manifold $F(2)$ as a subset of $\mathbf{P}^2 \times \mathbf{P}^2$, $PSL_2(\mathbf{C})$ acts properly and cocompactly on the dense domain of generic flags*

$$X_2 = F(2) \cap (\mathbf{P}^2 - Q_2) \times (\mathbf{P}^2 - Q_2).$$

The quotient $\mathrm{Sym}_2(PSL_2(\mathbf{C}))\backslash X_2$ is a point.

For any hyperbolic and orientable 3-manifold M^3, $\rho_2(\pi_1(M^3))\backslash X_2$ is a sphere bundle over M^3, obtained by quotienting out its frame bundle by $\Sigma_3 \ltimes (\mathbf{Z}/2\mathbf{Z})^3$. In particular it is the trivial sphere bundle.

This theorem tells that X_2 are the flags generic to Q_2 and its dual, see Figure 1.

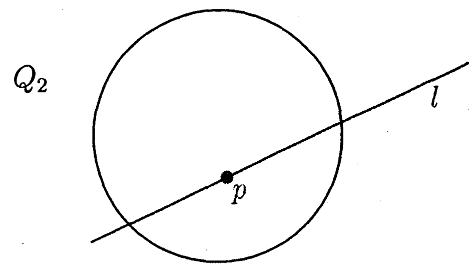

FIGURE 1. A generic flag: p does not belong to Q_2 and l is not tangent to Q_2.

To prove Theorem 2.3, we need the interpretation of Sym_2 as the adjoint representation. Let $\mathfrak{sl}_2(\mathbf{C})$ denote the Lie algebra. The following result is well known and it is a consequence of the uniqueness of irreducible representations of $PSL_2(\mathbf{C})$ in each dimension.

Proposition 2.4. *The adjoint action of $PSL_2(\mathbf{C})$ on $\mathfrak{sl}_2(\mathbf{C}) \cong \mathbf{C}^3$ is equivalent to Sym_2. Moreover it preserves the Killing form $B : \mathfrak{sl}_2(\mathbf{C}) \times \mathfrak{sl}_2(\mathbf{C}) \to \mathbf{C}$ and it defines an isomorphism $PSL_2(\mathbf{C}) \cong SO(3, \mathbf{C})$. The isomorphism maps the rational normal curve Q_2 to the zero set of the Killing form as a quadric $\{x \in \mathfrak{sl}_2(\mathbf{C}) \mid B(x,x) = 0\}$.*

Now we want to exploit the fact that $PSL_2(\mathbf{C})$ is the group of orientation preserving isometries of hyperbolic space. Let

$$P(\mathfrak{sl}_2(\mathbf{C})) \cong \mathbf{P}^2$$

denote the projective space on the Lie algebra. In particular, a point in $P(\mathfrak{sl}_2(\mathbf{C}))$ is a line in $\mathfrak{sl}_2(\mathbf{C})$ to which one can associate a one parameter group.

The following is straightforward.

Lemma 2.5. *For $x \in P(\mathfrak{sl}_2(\mathbf{C}))$, the one-parameter group of isometries*

$$\{\exp(\lambda x) \mid \lambda \in \mathbf{C}\}$$

is parabolic if $B(x,x) = 0$ and loxodromic if $B(x,x) \neq 0$.

By mapping a loxodromic one-parameter group to its invariant geodesic, we get:

Corollary 2.6. *There is a natural homeomorphism between*

$$P(\{x \in \mathfrak{sl}_2(\mathbf{C}) \mid B(x,x) \neq 0\})$$

and the set of unoriented geodesics of $\mathbf{H}^3$.

Recall that the boundary at infinity $\partial_\infty \mathbf{H}^3$ is equivalent to $\mathbf{P}^1$. Considering the end-points of geodesics, this corollary gives a homeomorphism

$$P(\{x \in \mathfrak{sl}_2(\mathbf{C}) \mid B(x,x) \neq 0\}) \cong (\partial_\infty \mathbf{H}^3 \times \partial_\infty \mathbf{H}^3 - \Delta)/\Sigma_2,$$

where Σ_2 is the permutation group of two elements and Δ the diagonal. This homeomorphism extends continuously to an homeomorphism

$$P(\{x \in \mathfrak{sl}_2(\mathbf{C}) \mid B(x,x) = 0\}) \cong \partial_\infty \mathbf{H}^3,$$

that maps a parabolic group of isometries to its invariant point at infinity. More precisely, we have the following definition:

Definition 2.7. The space of unoriented (and possibly degenerate) geodesics is

$$\mathcal{G}(\mathbf{H}^3) = (\partial_\infty \mathbf{H}^3 \times \partial_\infty \mathbf{H}^3)/\Sigma_2.$$

Corollary 2.8. *There is a natural homeomorphism*

$$\mathcal{G}(\mathbf{H}^3) \cong P(\mathfrak{sl}_2(\mathbf{C}))$$

which is $PSL_2(\mathbf{C})$-equivariant and that maps the degenerate geodesics $\partial_\infty \mathbf{H}^3 \subset \mathcal{G}(\mathbf{H}^3)$ to $Q_2 = P(\{x \in \mathfrak{sl}_2(\mathbf{C}) \mid B(x,x) = 0\})$.

The previous corollary gives already a geometric interpretation of points in $P(\mathfrak{sl}_2(\mathbf{C}))$. We aim to extend it to the flag manifold, in particular to the dual of $P(\mathfrak{sl}_2(\mathbf{C}))$, of course by means of the Killing form.

Namely, for each $x \in P(\mathfrak{sl}_2(\mathbf{C}))$, its B-orthogonal $x^\perp$ is a projective line in $P(\mathfrak{sl}_2(\mathbf{C}))$, and since B is nondegenerate this defines an isomorphism between $P(\mathfrak{sl}_2(\mathbf{C}))$ and its dual.

Lemma 2.9. *Given $l \in P(\mathfrak{sl}_2(\mathbf{C}))$, the following hold true.*

1. *If $B(l,l) = 0$ then $l^\perp$ is the subspace tangent to a group that fixes a point in $\partial_\infty \mathbf{H}^3$. In particular the geodesics corresponding to $l^\perp$ are all asymptotic to a fixed point in $\partial_\infty \mathbf{H}^3$.*
2. *If $B(l,l) \neq 0$ then the set geodesics corresponding to $l^\perp$ is a pencil of geodesics in $\mathbf{H}^3$ perpendicular to a fixed geodesic.*

Proof. When $B(l,l) = 0$, by transitivity of the action, we may assume that $l = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $l^\perp = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ and the exponential of $l^\perp$ is the set of all one parameter groups that fix the point with homogeneous coordinates $[1:0]$. Namely we obtain all geodesics asymptotic to $[1:0] \in \mathbf{P}^1 \cong \partial \mathbf{H}^3$.

When $B(l,l) \neq 0$, we assume that $l = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $l^\perp = \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Thus $l^\perp$ contains the parabolic elements $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, with respective fixed points in $\partial_\infty \mathbf{H}^3 \cong \mathbf{P}^1$ with homogeneous coordinates $[1:0]$ and $[0:1]$, as well as the loxodromic elements $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$, with $ab \neq 0$. Using the formulas of [26, Appendix] and the formalism of Fenchel's book [5], since these elements are orthogonal to l by the Killing form, the corresponding geodesics are orthogonal. Therefore we obtain the family of geodesics that are orthogonal to the geodesic with end-points $[1:0]$ and $[0:1]$ in $\mathbf{P}^2$. □

The dual of $P(\mathfrak{sl}_2(\mathbf{C}))$ and $\mathcal{G}(\mathbf{H}^3)$ may be identified to themselves, and we get:

Proposition 2.10. *The flag manifold is equivariantly homeomorphic to*

$$Z = \{(l_1, l_2) \in \mathcal{G}(\mathbf{H}^3) \times \mathcal{G}(\mathbf{H}^3) \mid l_1 \perp l_2\}.$$

This includes $\partial_\infty \mathbf{H}^3 \subset \mathcal{G}(\mathbf{H}^3)$ as degenerate geodesics, and the perpendicularity relation becomes being asymptotic.

Let $Z_0 \subset Z$ be the *nondegenerate* subset of Z, namely

$$Z_0 = Z \cap \left((\mathcal{G}(\mathbf{H}^3) - \partial_\infty \mathbf{H}^3) \times (\mathcal{G}(\mathbf{H}^3) - \partial_\infty \mathbf{H}^3) \right).$$

Remark 2.11. The set Z_0 is equivariantly homeomorphic to $\mathcal{F}(\mathbf{H}^3)/(\Sigma_3 \rtimes (\mathbf{Z}/2)^3)$, where $\mathcal{F}(\mathbf{H}^3)$ is the frame bundle of $\mathbf{H}^3$, Σ_3 acts by permutation of the vectors and $(\mathbf{Z}/2)^3$ by changes of sign of the vectors.

To prove Theorem 2.3, notice that $PSL_2(\mathbf{C})$ acts properly and cocompactly on the frame bundle $\mathcal{F}(\mathbf{H}^3)$, hence it acts properly and cocompactly on Z_0, the set or pairs of geodesics in $\mathbf{H}^3$ that are perpendicular. In addition, viewing the flag manifold $F(2)$ as a subset of $\mathbf{P}^1 \times \mathbf{P}^1$, $\mathrm{Sym}_2(PSL_2(\mathbf{C}))$ acts properly and cocompactly the dense domain $X_2 = F(2) \cap (B \neq 0)^2 \cong Z_0$.

The quotient $\mathrm{Sym}_2(PSL_2(\mathbf{C}))\backslash X_2$ is a point. For any hyperbolic orientable 3-manifold M^3, $\rho_2(\pi_1(M^3))\backslash X_2$ is a sphere bundle over M^3, obtained by quotienting out its frame bundle by $\Sigma_3 \ltimes (\mathbf{Z}/2)^3$. In particular it is the trivial sphere bundle.

This concludes the proof of Theorem 2.3.

3. The action of Sym_n for $n > 2$ and hyperbolic geometry

Recall that $\mathrm{Sym}_n(SL_2(\mathbf{C}))$ acts on the space homogeneous polynomials of $\mathbf{C}[X,Y]$ of degree n, that we denote by $\mathbf{C}_n[X,Y]$. We look for a domain in $\mathbf{P}^n = P(\mathbf{C}_n[X,Y])$ where the action is proper and cocompact.

We also recall the Veronese embedding (2)

$$\begin{array}{ccc} \mathbf{P}^1 & \to & \mathbf{P}^n \\ (a:b) & \mapsto & (aX+bY)^n \end{array} \tag{3}$$

with image Q_n, the rational normal curve.

Finally recall that the k-osculating variety to Q_n is the set of projective k-planes that are k-osculating to Q and it is denoted by $Osc_k(Q_n)$.

To prove Theorem 1.4, we fist show that the action $\mathrm{Sym}_n(PSL_2(\mathbf{C}))$ is proper on

$$X_n = \mathbf{P}^n - Osc_{[n/2]}(Q_n).$$

We also show that it is cocompact for n odd, and has a natural one point compactification when n is even. Naturality shall become clear from the proof.

In Section 4 we shall discuss the point of view of Mumford using Geometric Invariant Theory [21], and later the one of Deligne and Mostow [4]. In this section we follow an approach that uses mainly hyperbolic geometry. First we need to relate this action with the action on configurations of $\partial_\infty(\mathbf{H}^3) \cong \mathbf{P}^1$.

Definition 3.1. The space of *unordered configurations* of n points in the projective line $\mathbf{P}^1$ is

$$\mathrm{Conf}_n(\mathbf{P}^1) = (\mathbf{P}^1)^n/\Sigma_n,$$

where Σ_n denotes the permutation group.

To a polynomial in $\mathbf{C}_n[X,Y]$ we associate its n (unordered) roots in $\mathbf{P}^1$, hence we have an equivariant isomorphism:

$$\mathbf{P}^n \cong \mathrm{Conf}_n(\mathbf{P}^1) = (\mathbf{P}^1)^n/\Sigma_n \tag{4}$$

where $PSL_2(\mathbf{C})$ acts diagonally on $(\mathbf{P}^1)^n$ and Σ_n is the permutation group on n elements.
Let $\Delta_k \subset \mathbf{P}^n/\Sigma_n$ denote the k-diagonal, namely the subset such that (at least) k of its components are equal.

Remark 3.2. The isomorphism (4) identifies $Osc_k(Q_n) \subset (\mathbf{P}^1)^n$ with $\Delta_{n-k} \subset (\mathbf{P}^1)^n/\Sigma_n$.

Given an ideal point $\xi \in \partial_\infty \mathbf{H}^3$ and a geodesic ray $r : [0, +\infty) \to \mathbf{H}^3$ asymptotic to ξ, $\lim_{t\to+\infty} r(t) = \xi$, for any $x \in \mathbf{H}^3$ the quantity $t - d(x, r(t))$ is strictly increasing on t, and bounded above by $d(r(0), x)$, by the triangle inequality. Hence, the limit

$$\lim_{t\to+\infty} d(x, r(t)) - t$$

exists. It defines a function on $x \in \mathbf{H}^3$ such that, up to some additive constant, depends only on the ideal point $\lim_{t\to+\infty} r(t) = \xi \in \partial_\infty \mathbf{H}^3$ (see for instance [2]).

Definition 3.3. The *Busemann function* centered at ξ is

$$b_\xi(x) = \lim_{t\to+\infty} d(x, r(t)) - t,$$

for any choice of ray $r : [0, +\infty) \to \mathbf{H}^3$ satisfying $r(+\infty) = \xi$.

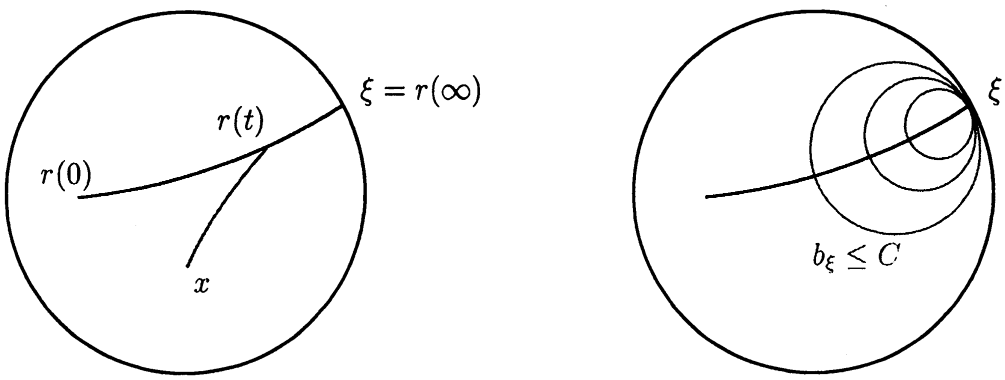

FIGURE 2. Definition of Busemann function (left) and its level subsets (right).

In the upper half space model for $\mathbf{H}^3$, $\{(z, t) \in \mathbf{C} \times \mathbf{R} \mid t > 0\}$ equipped with the metric

$$\frac{d|z|^2 + dt^2}{t^2},$$

and with boundary at infinity $\partial_\infty \mathbf{H}^3 \cong \mathbf{C} \cup \{\infty\}$, the Busemann function centered at $\xi = \infty$ is, up to some additive constant,

$$b_\infty(z, t) = -\log t. \tag{5}$$

Then it is straightforward that b_ξ is convex, its level sets $b_\xi = c$ are horospheres centered at ξ, and its level subsets $b_\xi \leq c$ are horoballs.
Given an unordered configuration

$$C = \{\xi_1, \ldots, \xi_n\} \in \mathrm{Conf}_n(\mathbf{P}^1) \cong (\mathbf{P}^1)^n/\Sigma_n,$$

consider the sum of Busemann functions:

$$b_C = b_{\xi_1} + \cdots + b_{\xi_n} : \mathbf{H}^3 \to \mathbf{R},$$

which is again a function well defined up to some additive constant.

Lemma 3.4. *For $n \geq 3$ and $C \in \mathrm{Conf}_n(\mathbf{P}^1)$, the function b_C is proper (has compact sublevel sets) iff no point of C has multiplicity at least $n/2$.*

Proof. We first look at the example of a configuration consisting of two points. Let $\xi_-, \xi_+ \in \mathbf{H}^3$ be different points, Consider a geodesic $\gamma : (-\infty, +\infty) \to \mathbf{H}^3$ that satisfies $\gamma(\pm\infty) = \xi_\pm$. Then $b_{\xi_-} + b_{\xi_+}$ is constant (and attains its minimum) along γ. Even if bounded below, $b_{\xi_-} + b_{\xi_+}$ is not proper, as the sublevel sets are noncompact. In addition, since Busemann functions are Lipschitz, it is bounded above in the metric tubular neighbourhood $\mathcal{N}_r(\gamma) = \{x \in \mathbf{H}^3 \mid d(x,\gamma) \leq r\}$.

To prove one implication of the lemma, assume that a point in the configuration has multiplicity $k \geq n/2$. In particular $\xi_1 = \cdots = \xi_k$. If $k = n$, obviously $b_c = n\, b_{\chi_1}$ is not proper. Otherwise, $\xi_{k+1}, \ldots, \xi_n$ are $n-k \leq n/2$ points in the configuration different from ξ_1. Consider the geodesics $\overline{\xi_1\xi_{k+1}}, \ldots, \overline{\xi_1\xi_n}$. By the previous discussion, the function $b_{\xi_1} + b_{\xi_{k+1}}$ is not only constant on $\overline{\xi_1\xi_{k+1}}$ but it is also bounded on $\overline{\xi_1\xi_{k+j}}$ when approaching ξ_1, for $j = 1, \ldots, n-k$, because both $\overline{\xi_1\xi_{k+1}}$ and $\overline{\xi_1\xi_{k+j}}$ are are asymptotic to ξ_1. The function b_C is the sum of such pairs $b_{\xi_1} + b_{\xi_{k+j}}$, which are bounded on $\overline{\xi_1\xi_{k+1}}$ when approaching ξ_1, added to possibly some b_{ξ_1}, that converges to $-\infty$ when approaching ξ_1 along $\overline{\xi_1\xi_{k+1}}$. Hence it is not proper.

For the other implication, assume that that b_C is not proper: let x_n be a diverging sequence in $\mathbf{H}^3$ such that $b_C(\xi)(x_n)$ remains bounded above. We may assume that $x_n \to \eta \in \partial_\infty \mathbf{H}^3$. If $\eta \neq \xi_i$, then $b_{\xi_i}(x_n) \to +\infty$, therefore we may assume that $\eta = \xi_1$. Let k be the multiplicity of ξ_1, we claim that $k \geq n/2$. Notice that for $\xi_j \neq \xi_1$, $b_{\xi_1} + b_{\xi_j}$ is bounded below in the whole $\mathbf{H}^3$, hence if $k < n/2$, then $b_C(x_n)$ would decompose as the addition of terms $b_{\xi_1}(x_n) + b_{\xi_j}(x_n)$ bounded below and terms $b_{\xi_{2k+j}}(x_n)$ converging to $+\infty$. □

Lemma 3.5. *If C contains at least three different points, then b_C is strictly convex.*

Proof. It is straightforward from (5) that b_{ξ_i} is convex, and that the second derivative at the point $x \in \mathbf{H}^3$ only vanishes in the directions perpendicular to the ray $\overline{x\xi_i}$. If C has at least three different points, then there is no common perpendicular to the rays emanating from x to the points of C. □

Corollary 3.6. *If no point of C has multiplicity at least $n/2$, then b_C has a unique minimum in $\mathbf{H}^3$.*

Definition 3.7. When no point of C has multiplicity at least $n/2$, the unique point where minimum of b_C is reached is called the *barycenter* or *center of mass* of C and it is denoted by bar_C.

Thus we have an equivariant map

$$\mathbf{P}^n - Osc_{[n/2]}(Q_n) \overset{\text{roots}}{\cong} (\mathbf{P}^1)^n/\Sigma_n - \Delta_{[(n+1)/2]} \xrightarrow{\text{barycenter}} \mathbf{H}^3. \tag{6}$$

Here we have used that $[n/2] + [(n+1)/2] = n$ and Remark 3.2. Notice that $PSL_2(\mathbf{C})$ acts properly and cocompactly on $\mathbf{H}^3$, so this construction gives *properness* of the action on $\mathbf{P}^n - Osc_{[n/2]}(Q_n)$.

To study cocompactness, we must analyze the fibre of the barycenter map (6), equipped with the action of $SO(3,\mathbf{R})$, the stabilizer of a point in $\mathbf{H}^3$. To understand this fibre, look at the tangent vectors from the center of mass to the ideal points. They are unit vectors $v_1, \ldots, v_n$ and satisfy $v_1 + \cdots + v_n = 0$. Thus define:

Definition 3.8. Define the space of unordered configurations in the unit sphere $S^2 \subset \mathbf{R}^3$ with barycenter the origin:

$$\mathrm{Conf}_n^0(S^2) = \{(v_1, \ldots, v_n) \in S^2 \times \cdots \times S^2 \mid v_1 + \cdots + v_n = 0\}/\Sigma_n.$$

We call a configuration in $\mathrm{Conf}_n^0(S^2)$ *regular* if it is supported in at least three different vectors. The set of all regular configurations is denoted by

$$\mathrm{Conf}_n^0(S^2)^{reg} = \{C \in \mathrm{Conf}_n^0(S^2) \mid C \text{ is supported in at least three different vectors}\}.$$

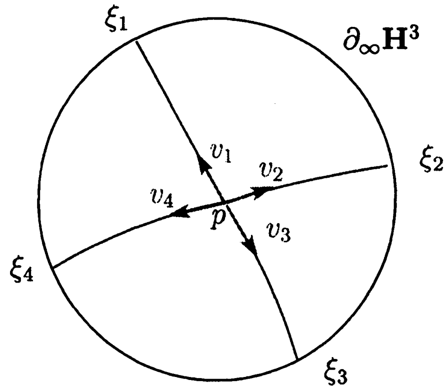

FIGURE 3. At the minimum the addition of the unit tangent vectors v_i vanishes.

Notice that for n odd, $\mathrm{Conf}_n^0(S^2)^{reg} = \mathrm{Conf}_n^0(S^2)$. For n even, the difference between $\mathrm{Conf}_n^0(S^2)^{reg}$ and $\mathrm{Conf}_n^0(S^2)$ is precisely the $SO(3)$-orbit of configurations supported on precisely two vectors, namely two opposite vectors that occur precisely $n/2$ times each.

Lemma 3.9. *The fibre of the barycenter map* (6) *is homeomorphic to* $\mathrm{Conf}_n^0(S^2)^{reg}$*, equipped with the action of* $SO(3)$.

For n odd, this proves cocompactness because $\mathrm{Conf}_n^0(S^2)^{reg} = \mathrm{Conf}_n^0(S^2)$ is compact, and so is $\mathrm{Conf}_n^0(S^2)^{reg}$. For n even $\mathrm{Conf}_n^0(S^2) - \mathrm{Conf}_n^0(S^2)^{reg}$ consists of a single orbit, thus $\mathrm{Conf}_n^0(S^2)/SO(3)$ is the one-point compactification of $\mathrm{Conf}_n^0(S^2)^{reg}/SO(3)$. Using Geometric Invariant Theory, we shall show in next section that $\mathrm{Conf}_n^0(S^2)/SO(3)$ is a projective variety smooth at $\mathrm{Conf}_n^0(S^2)^{reg}/SO(3)$.

If the configurations where ordered, they would correspond to polygons in $\mathbf{R}^3$ with sides of length one. This was studied by Kapovich and Millson in [14], where they view configurations as atomic measures. These ideas are further developed by Kapovich, Leeb and Millson in [13]. The idea of barycenter of measures is quite common and has many applications, as for instance the entropy rigidity of Besson, Courtois and Gallot [1].

4. The Geometric Invariant Theory approach

Here we apply the point of view of geometric invariant theory [21]. The actions of $PSL_2(\mathbf{C})$ on $\mathbf{P}^n$ and $(\mathbf{P}^1)^n$ are algebraic, so it makes sense to look at the quotients in geometric invariant theory. Geometric invariant theory provides Zariski open subsets $U \subset V$ of $\mathbf{P}^n$ and $(\mathbf{P}^1)^n$ that are $PSL_2(\mathbf{C})$-invariant and:

- A categorical quotient $\pi : V \to Z$. Namely this projection is constant on $PSL_2(\mathbf{C})$-orbits, and every algebraic map $V \to Y$ constant on $PSL_2(\mathbf{C})$-orbits factors though $V \to Z$.

- The projection $\pi : V \to Z$ restricts to a geometric quotient on U: $\pi(U)$ is open and the fibers of $\pi : \pi^{-1}(\pi(U)) \to U$ are orbits.

The choice of U and V is made by means of stability. We recall the following definition:

Definition 4.1. Let $V \subset \mathbf{C}^{n+1}$ be an affine cone, i.e. an algebraic variety such that if $x \in V$ then $\lambda x \in V$ $\forall \lambda \in \mathbf{C}$. Let G be a Lie group acting on V. A point $x \in V - \{0\}$ is called:

- *stable* if the orbit Gx is closed and x has finite stabilizer,
- *semistable* if 0 is not in the closure of the orbit Gx, and
- *unstable* if 0 is in the closure of the orbit Gx.

Let $P(V)^s$ and $P(V)^{ss}$ denote the subset of stable and semistable points, which are Zariski open. Geometric invariant theory provides the following:

Theorem 4.2 ([21], cf. [23], [27]). *Let Z be the projective variety whose graded algebra is $\mathbf{C}[V]^G$, the set of invariant functions of the algebra of V. Then:*

(1) *There is a projection $\pi : P(V)^{ss} \to Z$ that is the categorical quotient.*
(2) *The morphism $\pi : P(V)^{ss} \to Z$ is affine.*
(3) *The restriction to $P(V)^s$ is a geometric quotient.*

Remark 4.3. Notice that the projection on the set of semistable points $P(V)^{ss} \to PSL_2(\mathbf{C})\backslash P(V)^{ss}$ is the standard topological quotient, and that Z is a natural compactification.

Remark 4.4. Notice also that the topology of the orbits in V and in $P(V)$ may differ. In fact, for an stable point, its orbit in V is closed but possibly not in $P(V)$. However it is closed in $P(V)^{ss}$, the semistable part. The orbit of a semistable point maybe nonclosed in $P(V)^{ss}$, if not it accumulates to a closed orbit, which is unique in the fibre of π.

Back to our setting, $V = \mathbf{C}_n[X, Y]$, the space of homogeneous polynomials of degree n, and to a polynomial in $\mathbf{C}_n[X, Y]$ its roots in $\mathbf{P}^1$. Then we have:

Lemma 4.5. *A polynomial in $\mathbf{C}_n[X, Y]$ is stable iff all roots have multiplicity $< n/2$.*
It is semi-stable iff the multiplicities are $\leq n/2$.

We do not provide a proof of this lemma, which is stated in 1.7 of [22]. It is not difficult, by considering the Segre embedding of $(\mathbf{P}^1)^n$ in some projective space.

Let us try to understand this lemma in our setting. Notice first that it is coherent with the choice of domains of $\mathbf{P}^n$ we have made in the introduction. The action of $PSL_2(\mathbf{C})$ in the configuration space of roots can bring together different points, thus semistable orbits in $P(V)$ accumulate to unstable.

The discussion for semistability depends on the parity of n:

- Notice that when n is odd, semistable equals to stable, and this explains why we do not need to compactify in the odd case.
- When n is even the semistable but not stable polynomials have a root of multiplicity $n/2$. The orbits of such polynomials are nonclosed, and they accumulate to either unstable orbits or to an orbit with precisely two roots of multiplicity $n/2$. Thus all the semistable orbits project to a single point in the GIT quotient Z.

Using the isomorphism (4) and Remark 3.2, we get the following corollary of Lemma 4.5:

Corollary 4.6. *The stable and semistable sets are:*

$$(\mathbf{P}^n)^s = \mathbf{P}^n - Osc_{[n/2]}(Q_n) \qquad \text{and} \qquad (\mathbf{P}^n)^{ss} = \mathbf{P}^n - Osc_{[(n-1)/2]}(Q_n).$$

From the previous discussion we obtain:

Proposition 4.7. *The quotient $Y_n = PSL_2(\mathbf{C})\backslash(\mathbf{P}^n - Osc_{[n/2]}(Q_n))$ is*

- *a complex projective variety $\hat{Y}_n = Y_n$ of dimension $n-3$, for n odd;*
- *a complex projective variety $\hat{Y}_n$ of dimension $n-3$ minus one point, for n even.*

In Section 5 we will prove that $PSL_2(\mathbf{C})\backslash(\mathbf{P}^n - Osc_{[n/2]}(Q_n))$ is smooth, but the compactification for even $n \geq 6$ is singular.

5. Smoothness of the quotient

We shall show that Y_n has no singular point, and that, for even $n \geq 6$, the point $\hat{Y}_n - Y_n$ is a singular point. This uses essentially the methods of [12].

Since the stabilizer of a point in $P(V)^s$ is trivial, a straightforward application of Luna's slice theorem [15] gives:

Lemma 5.1. *All points of $Y_n = \pi(P(V)^s)$ are smooth.*

Lemma 5.2. *For $n \geq 6$, the point $\hat{Y}_n - Y_n = \pi(P(V)^{ss})$ is singular, but regular for $n = 4$.*

Proof. We look at the closed orbit corresponding to the completion, the polynomials of the form $m_1^{n/2} m_2^{n/2}$, for two different monomials m_1 and m_2. Since this is a single orbit, we may assume that the polynomial is $X^{n/2}Y^{n/2}$. The stabilizer of this orbit is the one-parameter group

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbf{C}^* \right\} \cong \mathbf{C}^*.$$

We work with homogeneous coordinates

$$[a_{-n/2}, a_{-n/2+1}, a_{-n/2+2}, \dots, a_{n/2}]$$

corresponding to the polynomial

$$\sum_{i=-n/2}^{n/2} a_i X^{n/2+i} Y^{n/2-i}.$$

In particular $X^{n/2}Y^{n/2}$ has coordinates $a_i = 0$ for $i \neq 0$ and $a_0 \neq 0$. To find a slice, fix first an affine chart determined by $a_0 = 1$, which is invariant under the action of the stabilizer.

We next determine the tangent space to the orbit of $X^{n/2}Y^{n/2}$. Consider the action of the infinitesimal isometries

$$h_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad h_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The infinitesimal action of h_+ does not change X and maps Y to $Y + \varepsilon X$, for infinitesimal ε, thus it maps

$$X^{n/2}Y^{n/2} \mapsto X^{n/2}Y^{n/2} + \frac{n}{2}\varepsilon X^{n/2+1}Y^{n/2-1} + O(\varepsilon^2).$$

Thus its tangent vector has coordinates $a_i = 0$ for $i \neq -1$ and $a_{-1} \neq 0$. Analogously, the tangent vector to the action of h_- has coordinates $a_i = 0$ for $i \neq 1$ and $a_1 \neq 0$. To have a transverse slice, define it by setting $a_0 = 1$ and $a_{-1} = a_1 = 0$:

$$S = \{[a_{-n/2} : a_{-n/2+1} : \cdots : a_{-2} : 0 : 1 : 0 : a_2 : \cdots : a_{n/2-1} : a_{n/2}] \mid a_i \in \mathbf{C}\} \cong \mathbf{C}^{n-3}$$

By construction S is transverse to the tangent space of the orbit at $X^{n/2}Y^{n/2}$ and invariant under the action of the stabilizer H. Hence it is the slice constructed in the proof of Luna's slice theorem [15]. It follows that the point in the quotient is singular iff S/H is singular at $a_i = 0$, for $i \neq 0$.

The next step will be to compute the quotient S/H, but we will have to distinguish different cases for n. We will use that the stabilizer is the one-parameter group that maps the coordinate a_i to $\lambda^{2i}a_i$.

We discuss first the case $n = 4$. Hence the coordinates are $(a_2, a_{-2}) \in \mathbf{C}^2$ and the functions invariant by H is the ring generated by the coordinate $x = a_{-2}a_2$. Hence $S/H \cong \mathbf{C}$ is smooth.

Next assume $n = 6$. The coordinates are $(a_3, a_2, a_{-2}, a_{-3}) \in \mathbf{C}^2$. Here the H-invariant functions are generated by

$$\begin{cases} x = a_2a_{-2} \\ y = a_3a_{-3} \\ z = a_2^3a_{-3}^2 \\ t = a_{-2}^3a_3^2. \end{cases}$$

They are not independent functions (the dimension of the quotient is 3), and satisfy the relation:

$$zt = x^3y^2, \tag{7}$$

which defines a hypersurface that is singular at the origin.

For larger n even, the H invariant functions are generated by

$$x_I = x_{i_1,i_2,\ldots,i_k} = a_{i_1}a_{i_2}\cdots a_{i_k},$$

satisfying $i_1 + i_2 + \cdots i_k = 0$. The equations are of the form

$$x_{I_1}x_{I_2}\cdots x_{I_r} = x_{J_1}x_{J_2}\cdots x_{J_s},$$

where the union of unordered set of indices are equal:

$$I_1 \cup I_2 \cup \cdots \cup I_r = J_1 \cup J_2 \cup \cdots \cup J_s.$$

Notice that $r, s \geq 2$ (otherwise this function is not required as generator), thus the derivative of the equation at the origin vanishes. Moreover, the set of equations is nonempty, because it always contains (7). Hence it is singular □

This finishes the proof of Theorem 1.4. Notice that in the proof we have obtained the following corollary.

Corollary 5.3. *The moduli space of unordered configurations of n unit vectors in $\mathbf{R}^3$ with trivial barycenter*

$$SO(3)\backslash \operatorname{Conf}_n^0(S^2)$$

is a complex projective variety which is smooth except at the point

$$(SO(3)\backslash \operatorname{Conf}_n^0(S^2)) - (SO(3)\backslash \operatorname{Conf}_n^0(S^2)^{reg})$$

for $n \geq 6$ even.

6. Low dimensional examples: $n = 3, 4, 5$

The goal of this section is to compute explicitly some quotients $Y_n = PSL_2(\mathbf{C})\backslash X_n$ for $n = 3$, 4, and 5.

6.1. **Case** $n = 3$. The space of ordered triples of different points is naturally isomorphic to the frame bundle of hyperbolic space. In our case, we consider unordered triples, so it is the quotient of the frame bundle by the permutation group acting on the vectors of the frame. In this case the osculating variety we remove is just the tangent variety, and the quotient

$$Y_3 = PSL_2(\mathbf{C})\backslash(\mathbf{P}^3 - Osc_1(Q_3)) \cong *$$

consists of just one point. The action of $PSL_2(\mathbf{C})$ is not effective, it has kernel Σ_3. Therefore

$$\pi_1(M^3)\backslash(\mathbf{P}^3 - Osc_1(Q_3)) \cong \pi_1(M^3)\backslash PSL_2(\mathbf{C})/\Sigma_3$$

is a quotient of the frame bundle over M^3 (the bundle of *unordered* frames).

6.2. **Case** $n = 4$. The space of ordered quadruples of different points has a natural function which is $PSL_2(\mathbf{C})$-invariant, the cross ratio:

$$[z_1 : z_2 : z_3 : z_4] = \frac{z_1 - z_3}{z_2 - z_3}\frac{z_2 - z_4}{z_1 - z_4}.$$

This defines a function on the set of different quadruples of $\mathbf{P}^1$ that extends when at most two points are equal:

$$\begin{array}{rcl} (\mathbf{P}^1)^4 - \Delta_3 & \to & \mathbf{P}^1 \\ (z_1, z_2, z_3, z_4) & \mapsto & [z_1 : z_2 : z_3 : z_4]. \end{array}$$

To get a function on the space of unordered configurations, we consider the action of three permutations that span the symmetric group on 4 elements:

$$\begin{array}{c} [z_2 : z_1 : z_3 : z_4] = [z_1 : z_2 : z_4 : z_3] = \frac{1}{[z_1:z_2:z_3:z_4]}, \\ [z_1 : z_3 : z_2 : z_4] = 1 - [z_1 : z_2 : z_3 : z_4]. \end{array} \tag{8}$$

Consider the branched covering $F : \mathbf{P}^1 \to \mathbf{P}^1$ of degree 6:

$$F(z) = \frac{z^6 - 3z^5 + 3z^4 - z^3 + 3z^2 - 3z + 1}{z^2(1-z)^2} = z^2 - z + \frac{3z^2 - 3z + 1}{z^2(1-z)^2},$$

It ramifies at $\infty \in \mathbf{C} \cup \{\infty\} = \mathbf{P}^1$ and satisfies $F^{-1}(\infty) = \{0, 1, \infty\}$. Moreover it is invariant by the transformations on the cross ratio (8)

$$F(z) = F(1 - z) = F(1/z),$$

It is then straightforward that

$$\begin{array}{rcl} (\mathbf{P}^1)^4/\Sigma_4 & \to & \mathbf{P}^1 \\ (z_1, z_2, z_3, z_4) & \mapsto & F([z_1 : z_2 : z_3 : z_4]) \end{array}$$

induces an isomorphism

$$\hat{Y}_4 = SL_2(\mathbf{C})\backslash(\mathbf{P}^4 - Osc_1(Q_4)) \cong \mathbf{P}^1.$$

In particular

$$\pi_1(M^3)\backslash(\mathbf{P}^4 - Osc_1(Q_4))$$

is a $\mathbf{P}^1$ bundle over the frame bundle of M^3.

6.3. **Case** $n = 5$. We start with the discussion of Deligne and Mostow [4] on the space of ordered configurations of 5 points, with at most two of them equal. Consider the map

$$\begin{array}{rcl} (\mathbf{P}^1)^5 - \Delta_3 & \to & \mathbf{P}^1 \times \mathbf{P}^1 \\ (z_1, z_2, z_3, z_4, z_5) & \mapsto & (\infty, 0, 1, \frac{1}{[z_1:z_2:z_3:z_4]}, \frac{1}{[z_1:z_2:z_3:z_5]}). \end{array}$$

It induces

$$\rho : PSL_2(\mathbf{C})\backslash((\mathbf{P}^1)^5 - \Delta_3) \to \mathbf{P}^1 \times \mathbf{P}^1.$$

The map ρ is birregular except at

$$L_{13} = \rho^{-1}(0,0), \qquad L_{12} = \rho^{-1}(1,1), \qquad L_{23} = \rho^{-1}(\infty,\infty).$$

Hence the quotient of the (ordered) configuration space $PSL_2(\mathbf{C})\backslash((\mathbf{P}^1)^5 - \Delta_2)$ is a blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at the three points $(0,0)$, $(1,1)$ and (∞,∞). Here L_{ij} corresponds to the coordinates i and j being equal. These are 10 lines in $PSL_2(\mathbf{C})\backslash((\mathbf{P}^1)^5 - \Delta_2)$, the three exceptional fibers ($\rho^{-1}(0,0)$, $\rho^{-1}(1,1)$, and $\rho^{-1}(\infty,\infty)$) and the ρ-lifts of seven lines in $\mathbf{P}^1 \times \mathbf{P}^1$:

$$x = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad y = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad x = y,$$

where $x = 1/[z_1 : z_2 : z_3 : z_4]$ and $y = 1/[z_1 : z_2 : z_3 : z_5]$.

To determine $PSL_2(\mathbf{C})\backslash X_5$ we consider the action of the permutation group Σ_5, namely:

$$PSL_2(\mathbf{C})\backslash X_5 \cong (\mathbf{P}^1 \times \mathbf{P}^1 \# 3\overline{\mathbf{P}^2})/\Sigma_5.$$

We already know that $PSL_2(\mathbf{C})\backslash X_5$ is a smooth complex projective surface. We need to argue that it is simply connected and then look at the homology and apply Freedman's theorem [6]. We describe the action of Σ_5. We look at permutations $(1i)$ of the first coordinate with the i-th coordinate, and the induced map on $\mathbf{P}^1 \times \mathbf{P}^1 \# 3\overline{\mathbf{P}^2}$, with a computation similar to the previous subsection. Notice that these permutations generate Σ_5. The induced maps are:

- The permutation (12) induces

$$\begin{cases} x \mapsto 1/x \\ y \mapsto 1/y \end{cases}.$$

- The permutation (13) induces

$$\begin{cases} x \mapsto \frac{x}{x-1} \\ y \mapsto \frac{y}{y-1} \end{cases}.$$

- The permutation (14) induces

$$\begin{cases} x \mapsto 1 - x \\ y \mapsto \frac{y(1-x)}{y-x} \end{cases}.$$

- The permutation (15) induces

$$\begin{cases} x \mapsto \frac{x(1-y)}{x-y} \\ y \mapsto 1 - y \end{cases}.$$

All these induced maps have fixed points. This implies that $PSL_2(\mathbf{C})\backslash X_5$ is simply-connected, because $\pi_1(PSL_2(\mathbf{C})\backslash X_5)$ is the quotient of the orbifold group, Σ_5, by the group generated by elements with fixed points, see for instance [11].

On the other hand Σ_5 obviously acts transitively on the ten lines l_{ij} defined by two coordinates being equal. Those lines generate the homology of $\mathbf{P}^1 \times \mathbf{P}^1 \# 3\overline{\mathbf{P}^2}$, hence the homology of the quotient has rank one, therefore:

$$Y_5 = PSL_2(\mathbf{C})\backslash(\mathbf{P}^5 - Osc_2(Q_5)) \cong \mathbf{P}^2.$$

Hence

$$\pi_1(M^3)\backslash(\mathbf{P}^5 - Osc_2(Q_5))$$

is a $\mathbf{P}^2$-bundle over the frame bundle of M^3.

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